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A RELATIONSHIP BETWEEN THE JONES AND KAUFFMAN POLYNOMIALS

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ABSTRACT. A simple relationship is presented between the Kauffman polynomial of a framed link L and the Jones polynomial of a derived link $\tilde{L}$. The link is $\tilde{L}$ obtained by splitting each component of L into two parallel strands, using the framing to determine the linking number of the strands. The relation is checked in several nontrivial examples, and a proof of the general result is given.

1. INTRODUCTION

Shortly after V. Jones discovered a new polynomial invariant $V_L(t)$ for oriented links [1], several generalizations were found. One of these was the two-variable Kauffman polynomial $F_L(a, x)$ [2]. It was quickly realized that this polynomial, when restricted to a one-dimensional subset of its parameter space, reproduces the Jones polynomial [3]. Specifically,

$$V_L(t) = F_L(t^{-3/4}, -(t^{-1/4} + t^{1/4})). \tag{1.1}$$

In this paper we introduce a new relationship between the Jones and Kauffman polynomials. It differs substantially from the relation just stated, because it relates the Jones and Kauffman polynomials of different links. It is easiest to state for a knot, so we will explain this case first.

Suppose K is a knot, that is, an embedding of S^1 in $\mathbb{R}^3$. Imagine now that this knot is constructed from a closed piece of string, and that the string is itself composed of two parallel, braided strands. As long as the strands are stuck together, we perceive a single closed loop. However, if the strands become unstuck and then separate, we obtain a link $\tilde{K}$ with two components. For example, Figure 1.1 shows the unknot and a link obtained from it by the procedure described. The main result in this paper is a relationship between the Kauffman polynomial of K and the Jones polynomial of $\tilde{K}$.

There is some ambiguity in the procedure described above. Firstly, we will be concerned throughout with oriented links, and so we must specify how the orientations of K and $\tilde{K}$ are related. We will always assume that the strands in $\tilde{K}$ are oriented so that when they are stuck together to make the knot K, their directions are parallel and agree with the orientation of K. The link $\tilde{U}$ in Figure 1.1 illustrates this.

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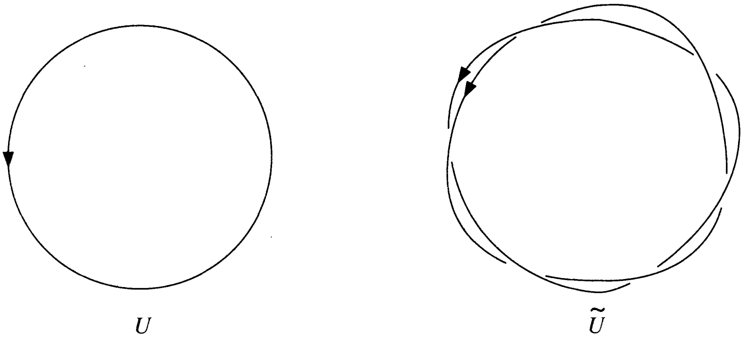


FIGURE 1.1. The unknot U and a link $\widetilde{U}$

FIGURE 1.2. Conventions for the linking number

Secondly, we must specify how the two strands in $\widetilde{K}$ are braided together to make the single loop of string. This braiding is completely described by an integer f, which is the Gauss linking number of the two strands. Recall that the linking number of two oriented knots K_1, K_2 is defined as follows [2]. In a regular projection of the link, we assign a number $\varepsilon(p) = \pm 1$ to each point p where K_1 and K_2 cross, according to the conventions in Figure 1.2. The linking number is then defined to be

$$w(K_1, K_2) = \frac{1}{2} \sum_p \varepsilon(p). \tag{1.2}$$

It is easy to check that $w(K_1, K_2)$ is invariant under the Reidemeister moves and hence is a link invariant. As an example, the linking number of the strands in $\widetilde{U}$ in Figure 1.1 is $f = 3$.

Once we specify K and the linking number f, this uniquely determines $\widetilde{K}$. Therefore it is natural to think of K as a framed knot, and we will do so from now on. We will always assume that the framing is equal to the linking number of the strands in $\widetilde{K}$.

We can now state our first main theorem.

Theorem 1. *Let K be an oriented, framed knot, with framing number f. Let $\widetilde{K}$ be the two-component link obtained as described above. Then*

$$t^f(1 + t + t^{-1})F_K(it^{-2}, i(t - t^{-1})) = -(t^{1/2} + t^{-1/2})V_{\widetilde{K}}(t) - t^{3f}. \tag{1.3}$$

The proof of Theorem 1 is given in §3. Before proceeding, we present some examples which test (1.3). First of all, when K is the unknot U, we have the situation depicted in Figure 1.1. It is easy to compute the Jones polynomial for

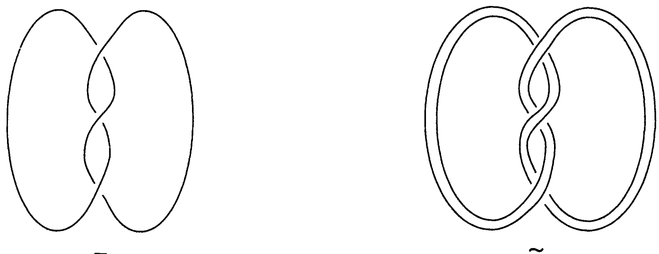


FIGURE 1.3. The trefoil and a derived link

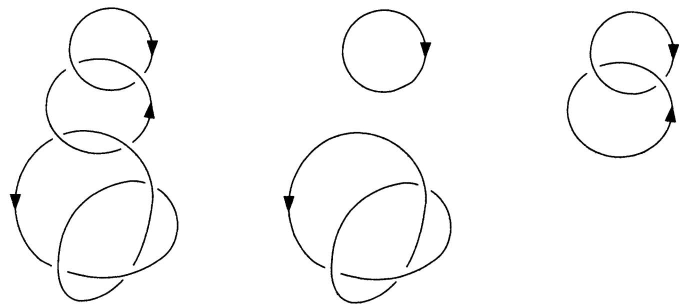


FIGURE 1.4. A link and two of its sublinks

$\widetilde{U}$ with an arbitrary framing f, and it is given by

$$V_{\widetilde{U}}(t) = -(t^{1/2} + t^{-1/2})^{-1}(t^{3f} + t^{f+1} + t^f + t^{f-1}). \tag{1.4}$$

Since U is the unknot, $F_U = 1$, and we see that (1.3) is verified.

A more formidable test is provided by the example in Figure 1.3, where T is the trefoil. In this case, the Kauffman polynomial of K is easily computed, and we obtain

$$F_T(it^{-2}, i(t - t^{-1})) = t^{11} - t^{10} - t^9 + t^8 - t^7 + t^5 + t^2. \tag{1.5}$$

The Jones polynomial of $\widetilde{T}$ can be computed with some difficulty, and we obtain

$$-(t^{1/2} + t^{-1/2})V_{\widetilde{T}}(t) = t^{15} - t^{13} - t^{12} - t^{11} + t^9 + t^8 + t^7 + t^6 + t^5 + t^4. \tag{1.6}$$

From Figure 1.3 we can read off the linking number and we get $f = 3$. The reader can check that (1.3) is satisfied when (1.5) and (1.6) are substituted in.

We will now describe the corresponding result for a general framed link L. Once again, we want to separate each component of L into two parallel strands. So we will need to specify an integer for each component of L, which will equal the linking number of the two strands in that component. This is equivalent to saying that L is a framed link.

Suppose the components of L are $\{C_1, \dots, C_n\}$, and $f(C_k)$ is the framing of the component C_k. We will write $w(C_k, C_l)$ for the linking number of the components C_k and C_l. Furthermore, we will want to consider sublinks of L, obtained by removing from L some of its components, without disturbing the others. Some examples are shown in Figure 1.4.

Given a subset J of $\{C_1, \ldots, C_n\}$, we will write $J \subset L$ to indicate that J is a sublink of L. Given a sublink J, we define

$$f(J) = \sum_{k\,:\,C_k \in J} f(C_k), \tag{1.7}$$

$$n(J) = \sum_{\substack{k, l\,:\,C_k \in J \\ C_l \in J}} w(C_k, C_l). \tag{1.8}$$

We want to allow J to be the empty set, in which case (1.7) and (1.8) are zero. As usual, we will denote by $|J|$ the cardinality of J, and by J^c the complement of J. Finally, we adopt the convention that when $J = \varnothing$ is the empty set

$$V_\varnothing(t) = -(t^{1/2} + t^{-1/2})^{-1}. \tag{1.9}$$

Theorem 2. *Let L be a framed, oriented link. Then with the notation and conventions defined above.*

$$\begin{aligned}(1 &+ t + t^{-1})t^{f(L)-4n(L)}F_L(it^{-2}, i(t - t^{-1})) \\ &= (t^{1/2} + t^{-1/2})\sum_{J \subset L}(-1)^{|J|}t^{3f(J^c)-6n(J)}V_{\tilde{J}}(t).\end{aligned} \tag{1.10}$$

This result will be proved in §3. Notice that if L has n components, the right-hand side of (1.10) has 2^n terms. Also when n is one, Theorem 2 reduces to Theorem 1.

Once again, we present some test examples. When L is composed of k unlinked unknots, each with zero framing, the left-hand side of (1.10) is

$$(-1)^{k-1}(1 + t + t^{-1})^k. \tag{1.11}$$

The right-hand side of (1.10) is

$$\begin{aligned}(t^{1/2} &+ t^{-1/2})\sum_{J \subset L}(-1)^{|J|}(-1)(t^{1/2} + t^{-1/2})^{-1}(2 + t + t^{-1})^{|J|} \\ &= (-1)^{k+1}\sum_{J \subset L}(-1)^{|J^c|}(2 + t + t^{-1})^{|J|} \\ &= (-1)^{k+1}(2 + t + t^{-1} - 1)^k.\end{aligned} \tag{1.12}$$

A less trivial example is provided by the two-component link in Figure 1.5. We denote by C_1 and C_2 the left- and right-hand components of L, and then we have

$$f(C_1) = f(C_2) = 0, \qquad w(C_1, C_2) = 1. \tag{1.13}$$

The left-hand side of (1.10) is easily computed, and we get

$$(1 + t + t^{-1})t^{-4}F_L(it^{-2}, i(t - t^{-1})) = -(1 + t + t^{-1})t^{-4}(t^7 + t^4 + t). \tag{1.14}$$

The right-hand side of (1.10) has four terms

$$(t^{1/2} + t^{-1/2})\{t^{-6}V_{\tilde{L}}(t) - V_{\tilde{C}_1}(t) - V_{\tilde{C}_2}(t) + V_\varnothing(t)\}. \tag{1.15}$$

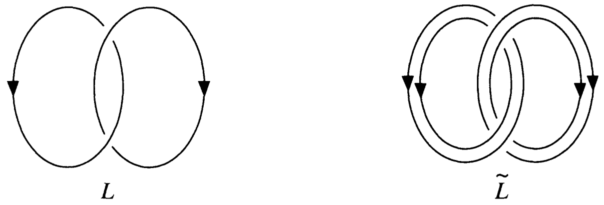


FIGURE 1.5. Two linked circles and a derived link

By explicit computation, we get

$$(1.16)\qquad \begin{gathered}(t^{1/2}+t^{-1/2})V_{\widetilde{L}}(t) = -(t^{10}+t^9+t^8+3t^7+4t^6+3t^5+t^4+t^3+t^2)\,,\\ V_{\widetilde{C}_1}(t) = V_{\widetilde{C}_2}(t) = -(t^{1/2}+t^{-1/2})\,,\quad V_{\varnothing}(t) = -(t^{1/2}+t^{-1/2})^{-1}.\end{gathered}$$

The reader can check that when (1.16) is substituted into (1.15), we reproduce (1.14).

In terms of utility, Theorem 2 is back to front. It expresses a specialization of the Kauffman polynomial of a link L in terms of the Jones polynomials of more complicated links. However, the relation can be inverted to express $V_{\widetilde{L}}(t)$ in terms of the Kauffman polynomials of L, and its sublinks. This time we use the convention that

$$(1.17)\qquad F_{\varnothing}(it^{-2}\,,\, i(t-t^{-1})) = -(1+t+t^{-1})^{-1}.$$

Theorem 3.

$$(1.18)\qquad \begin{aligned}&(t^{1/2}+t^{-1/2})t^{-6n(L)}V_{\widetilde{L}}(t)\\ &\quad = (1+t+t^{-1})\sum_{J\subset L}(-1)^{|J|}t^{f(J)+3f(J^c)-4n(J)}F_J(it^{-2}\,,\, i(t-t^{-1})).\end{aligned}$$

The proofs of Theorems 1, 2, 3 are presented in §3. They are completely elementary in the sense that they use only the crossing relations which serve as definitions of the Jones and Kauffman polynomials. A good review of these relations can be found in [2, 4].

The author was led to search for relations of this kind after studying the representations of the braid group which lead to link invariants [5–9]. In particular, these representations arise in conformal field theory [10, 11]. The idea of "fusion" in conformal field theory is closely related to the process of going from a two-stranded link $\widetilde{K}$ to a knot K, as described at the beginning of this section.

2. CROSSING RELATIONS FOR THE JONES POLYNOMIAL

In this section we derive new crossing relations for the Jones polynomial. These are the essential ingredients in the proof of Theorem 2.

We begin by recalling the usual crossing relations for the Jones polynomial. These include the original relation stated by Jones [1], as well as later additions by Birman and Kanenobu [12]. Suppose three links L_+, L_- and L_0 differ only at one crossing, in the manner indicated in Figure 2.1. Then the corresponding link polynomials satisfy the relation

$$(2.1)\qquad V_{L_+} = t^2V_{L_-} - t^{1/2}(1-t)V_{L_0}.$$

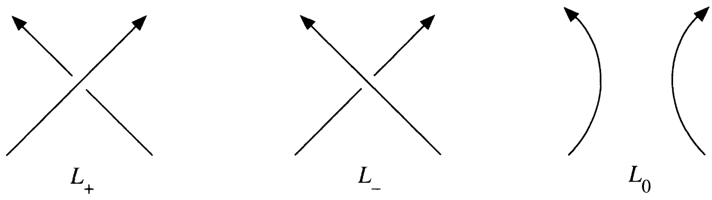


FIGURE 2.1. Crossings for the Jones polynomial

FIGURE 2.2. L_∞ for (a) Case 1 and (b) Case 2

Together with the relation $V_U = 1$ for the unknot U, this suffices to determine $V_L(t)$ for any link L. As Birman pointed out, there is another relation satisfied by $V_L(t)$. This divides into two cases, as follows.

Case 1. The two links in L_+ shown in Figure 2.1 belong to the same component of L_+. In this case, we introduce a new link L_∞ as shown in Figure 2.2(a).

Notice that the orientation of some links have been changed. Also the two lines in L_0 shown in Figure 2.1 belong to different components now. Let λ be the linking number of the right-hand component with the rest of L_0. Then

$$V_{L_+} = tV_{L_-} + (1-t)t^{3\lambda}V_{L_\infty}. \tag{2.2}$$

Case 2. The two lines in L_+ shown in Figure 2.1 belong to different components. In this case, we define the link L_∞ as indicated in Figure 2.2(b). Also, let μ be the linking number of the bottom right to top left component of L_+ shown in Figure 2.1 with the rest of L_+. Then

$$V_{L_+} = tV_{L_-} + (1-t)t^{3(\mu-1/2)}V_{L_\infty}. \tag{2.3}$$

Using (2.1) and (2.2), we can deduce the following further relations for Case 1.

$$V_{L_+} = -t^{1/2}V_{L_0} - t^{3\lambda+1}V_{L_\infty}, \quad V_{L_-} = -t^{-1/2}V_{L_0} - t^{3\lambda-1}V_{L_\infty}. \tag{2.4}$$

Similarly in Case 2 we have

$$V_{L_+} = -t^{1/2}V_{L_0} - t^{3\mu-1/2}V_{L_\infty}, \quad V_{L_-} = -t^{-1/2}V_{L_0} - t^{3\mu-5/2}V_{L_\infty}. \tag{2.5}$$

Our new relations will concern crossings of pairs of lines, which arise when a link L is replaced by $\widetilde{L}$. The diagrams corresponding to Figure 2.1 are shown in Figure 2.3. The relations satisfied by $V_{\widetilde{L}_+}(t)$ and $V_{\widetilde{L}_-}(t)$ separate into the two cases described previously. We will state and prove the result for Case 1 first, and then the result for Case 2.

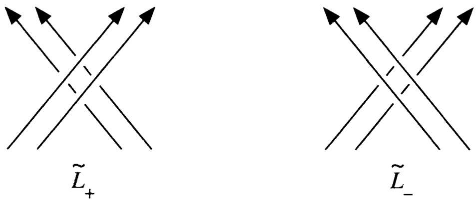


FIGURE 2.3. Crossings in $\widetilde{L}$

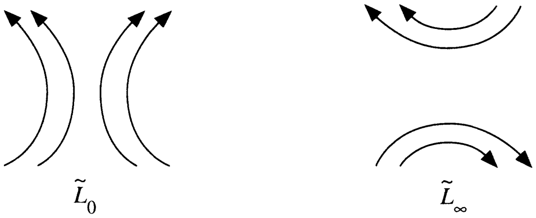


FIGURE 2.4. Crossings in $\widetilde{L}_0$, $\widetilde{L}_\infty$ for Case 1

Case 1. In order to simplify the notation, we will let C_+ denote the component of L_+ which contains the line segments shown in Figure 2.1, and let M denote the remainder of L_+. This allows us to write $L_+ = M \cup C_+$. Similarly we write

$$L_- = M \cup C_-, \qquad L_\infty = M \cup C_\infty.$$

The link L_0 has two components corresponding to C_+. We call these C_L and C_R, for the left- and right-hand sides shown in Figure 2.1. Then

$$L_0 = M \cup C_L \cup C_R.$$

The corresponding crossings in $\widetilde{L}_0$ and $\widetilde{L}_\infty$ are shown in Figure 2.4. In addition, we will also encounter links in which one of the components C_L or C_R has been removed. The doubled links will be denoted $\widetilde{L}_L$ and $\widetilde{L}_R$ respectively. When both components C_L and C_R are removed, the doubled link is just $\widetilde{M}$.

In order to state our results, we also need to specify some linking numbers and framings. We define

$$\begin{aligned} \alpha &= w(C_L, C_R), \\ w_L(M) &= w(C_L, M) = \sum_{k\,:\,C_k \in M} w(C_L, C_k), \qquad (2.6) \\ w_R(M) &= w(C_R, M). \end{aligned}$$

Also we define the framings on C_L and C_R to be

$$f(C_L) = f_L, \qquad f(C_R) = f_R. \qquad (2.7)$$

Using these values, we can compute the framings of C_+, C_- and C_∞. The results are

$$\begin{gathered} f(C_+) = f_L + f_R + 2\alpha + 1, \quad f(C_-) = f_L + f_R + 2\alpha - 1, \\ f(C_\infty) = f_L + f_R - 2\alpha. \end{gathered} \qquad (2.8)$$

$\widetilde{L}_0$ $\widetilde{L}_\infty$

FIGURE 2.5. Crossings in $\widetilde{L}_0$, $\widetilde{L}_\infty$ for Case 2

Our first result is the following:

Proposition 2.1.

$$t^{-3}V_{\widetilde{L}_+}(t) - t^3 V_{\widetilde{L}_-}(t) = (t - t^{-1})\{t^{12\alpha+12w_R(M)}V_{\widetilde{L}_\infty}(t) - V_{\widetilde{L}_0}(t) + t^{3f_R+6\alpha+6w_R(M)}V_{\widetilde{L}_L}(t) + t^{3f_L+6\alpha+6w_L(M)}V_{\widetilde{L}_R}(t) - 2t^{3f_L+3f_R+6\alpha+6w_L(M)+6w_R(M)}V_{\widetilde{M}}(t)\}. \tag{2.9}$$

Proof. The proof follows by a sequence of applications of the crossing relations (2.1)–(2.5). We will indicate the steps in the computation by using diagrams to represent link polynomials. These diagrams will be called $D_1, D_2, \dots, D_{15}$ in the following equations. They are shown in Figure 2.6. First we apply (2.1) and (2.2) to $V_{\widetilde{L}_+}(t)$ and $V_{\widetilde{L}_-}(t)$.

$$D_1 = t^3 D_2 + (1-t)t^{7/2}t^{3f_L+3f_R+6\alpha+6w_L(M)+6w_R(M)}D_3 - (1-t)t^{1/2}D_4. \tag{2.10}$$

$$D_5 = t^{-3}D_2 - (1-t)t^{-9/2}t^{3f_L+3f_R+6\alpha+6w_L(M)+6w_R(M)}D_6 + (1-t)t^{-3/2}D_7. \tag{2.11}$$

Combining the expressions in (2.10) and (2.11) we get

$$\begin{aligned} t^{-3}V_{\widetilde{L}_+(t)-t^3V_{\widetilde{L}_-}}(t) = {} & (1-t)t^{1/2}t^{3f_L+3f_R+6\alpha+6w_L(M)+6w_R(M)}D_3 \\ & + (1-t)t^{-3/2}t^{3f_L+3f_R+6\alpha+6w_L(M)+6w_R(M)}D_6 \\ & - (1-t)t^{-5/2}D_4 - (1-t)t^{3/2}D_7. \end{aligned} \tag{2.12}$$

Now we use (2.4) twice on each term in (2.12). As an example, we write out the result of this for the first diagram on the right-hand side of (2.12).

$$D_3 = t^{-3/2}t^{-3f_L-6w_L(M)}D_8 - t^{-1/2}t^{-3f_L-6w_L(M)}D_9 - t^{-3/2}t^{-3f_R-6w_R(M)}D_{10} + t^{-3/2}(1+t). \tag{2.13}$$

In the second term on the right side of (2.13), the component C_R has been removed completely from the link. The diagram therefore represents $V_{\widetilde{L}_L}(t)$. Similarly C_L has been removed in the third term, and both C_L and C_R have been removed in the fourth term. The corresponding polynomials are $V_{\widetilde{L}_R}(t)$ and $V_{\widetilde{M}}(t)$, respectively.

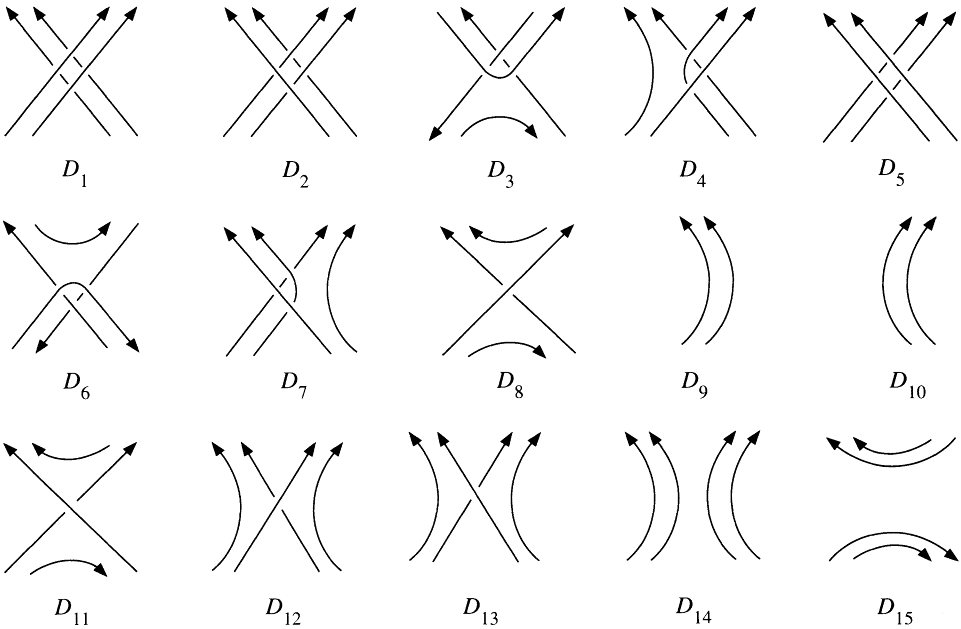


FIGURE 2.6. Diagrams for link polynomials in Case 1

Performing the same expansion for the other terms in (2.12) and combining the results, we get the following:

(2.14)

$$\begin{aligned} & t^{-3}V_{\widetilde{L}_+}(t) - t^3 V_{\widetilde{L}_-}(t) \\ & \quad = (t - t^{-1})\{t^{3f_R+6\alpha+6w_R(M)} V_{\widetilde{L}_L}(t) + t^{3f_L+6\alpha+6w_L(M)} V_{\widetilde{L}_R}(t)\} \\ & \quad\quad + (1-t)t^{-1/2}\{t^{3f_R+6\alpha+6w_R(M)}(t^{-1/2}D_8 + t^{1/2}D_{11}) - (t^{-1}D_{12} + tD_{13})\} \\ & \quad\quad - 2(t - t^{-1})t^{3f_L+3f_R+6\alpha+6w_L(M)+6w_R(M)} V_{\widetilde{M}}(t). \end{aligned}$$

A final application of (2.4) gives the following identity:

$$\begin{aligned} & t^{3f_R+6\alpha+6w_R(M)}(t^{-1/2}D_8 + t^{1/2}D_{11}) - (t^{-1}D_{12} + tD_{13}) \\ & \quad = (t^{1/2} + t^{-1/2})(D_{14} - t^{12\alpha+12w_R(M)}D_{15}). \end{aligned} \tag{2.15}$$

The first and second diagrams on the right side of (2.15) represent $V_{\widetilde{L}_0}$ and $V_{\widetilde{L}_\infty}$ respectively. Substituting (2.15) into (2.14), we obtain the result (2.9). Q.E.D.

Case 2. In this case, the link L_+ shown in Figure 2.1 has two components displayed. We will denote by C_1 the component going from the bottom right to the top left, and by C_2 the other component. Once again, the remainder of L_+ is denoted by M, so we have

$$L_+ = M \cup C_1 \cup C_2.$$

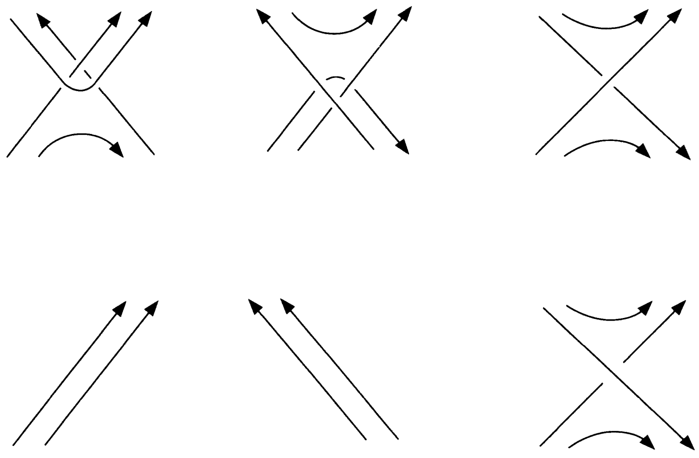


FIGURE 2.7. Diagrams for link polynomials in Case 2

The doubled links $\widetilde{L}_0$ and $\widetilde{L}_\infty$ are shown in Figure 2.5. We will need the following linking numbers and framings:

$$\begin{aligned} \beta &= w(C_1, C_2), \\ w_1(M) &= w(C_1, M), \qquad w_2(M) = w(C_2, M), \\ f_1 &= f(C_1), \qquad f_2 = f(C_2). \end{aligned} \tag{2.16}$$

Computing the framings in C_0 and C_∞, we get

$$f(C_0) = f_1 + f_2 + 2\beta - 1, \qquad f(C_\infty) = f_1 + f_2 - 2\beta + 1. \tag{2.17}$$

We now have our second result.

Proposition 2.2.

$$t^{-3} V_{\widetilde{L}_+}(t) - t^3 V_{\widetilde{L}_-}(t) = (t - t^{-1})\{t^{12\beta-6+12w_1(M)} V_{\widetilde{L}_\infty}(t) - V_{\widetilde{L}_0}(t)\}. \tag{2.18}$$

Proof. Once again, the proof involves application of the identities (2.1), (2.3), (2.5). We present the main steps in the computation below. There are some new diagrams $D_{16}, \dots, D_{21}$ which are shown in Figure 2.7.

$$D_1 = t^3 D_2 + (1-t) t^{-5/2} t^{3f_1+6\beta+6w_1(M)} D_{16} - (1-t) t^{1/2} D_4. \tag{2.19}$$

$$D_5 = t^{-3} D_2 - (1-t) t^{-9/2} t^{3f_1+6\beta+6w_1(M)} D_{17} + (1-t) t^{-3/2} D_7. \tag{2.20}$$

As an example, we give the expansion of the second term in (2.19) below.

$$\begin{aligned} D_{16} = {} & t^{-3f_1+6\beta+6w_1(M)} D_{18} - t^{3/2} D_{19} \\ & - t^{5/2} t^{-3f_1+3f_2-6w_1(M)+6w_2(M)} D_{20} + t^{3/2}(1+t) t^{3f_2+6w_2(M)}. \end{aligned} \tag{2.21}$$

In the second term on the right in (2.21), the component C_1 has been removed from L_+. Similarly in the third term, C_2 has been removed, and both C_1 and C_2 are absent in the fourth term.

Making similar expansions for the other terms in (2.19) and (2.20), we deduce the following relation:

$$\begin{aligned} t^{-3} V_{\widetilde{L}_+}(t) - t^3 V_{\widetilde{L}_-}(t) = (1-t) t^{-1/2} \{ & t^{12\beta-6+12w_1(M)} (t^{-1} D_{21} + t D_{18}) \\ & - (t^{-1} D_{12} + t D_{13}) \}. \end{aligned} \tag{2.22}$$

One last application of (2.5) to each term in (2.22) produces the desired result (2.18). Q.E.D.

3. Proofs of theorems

Since Theorem 1 is a special case of Theorem 2, we need only establish the result stated in Theorem 2. We do this by using the results from §2 to analyze the right-hand side of (1.10), and prove that it satisfies the crossing relations of the Kauffman polynomial. In §2, we stated the usual crossing relations for the Jones polynomial. The corresponding relations for the Kauffman polynomial separate into the same two cases [3].

Case 1. With the same notation as in §2,

$$aF_{L_+}(a, x) + a^{-1}F_{L_-}(a, x) = x\{F_{L_0}(a, x) + a^{-4\lambda}F_{L_\infty}(a, x)\}. \tag{3.1}$$

Case 2. Again with the same notation,

$$aF_{L_+}(a, x) + a^{-1}F_{L_-}(a, x) = x\{F_{L_0}(a, x) + a^{-4\mu+2}F_{L_\infty}(a, x)\}. \tag{3.2}$$

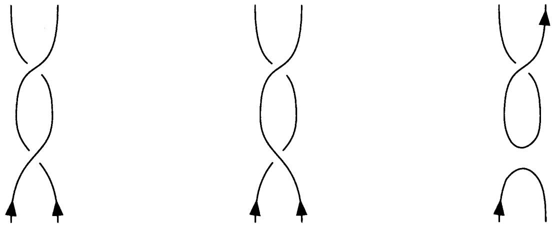
As mentioned before, we will prove Theorem 2 by showing that the expression (1.10) satisfies the relations (3.1) and (3.2) which define the Kauffman polynomial. As a first step, we will show that the expression (1.10) transforms correctly when the framing of the link is changed. For this, it is sufficient to consider what happens when the framing of one component in L changes by one. We will list the components of L as $\{C_1, \dots, C_n\}$, and then $f(C_k)$ is the framing of the component C_k. Also $w(C_k, C_l)$ is the linking number of components C_k and C_l. If M is a sublink of L which does not contain C_k, we will write

$$w(C_k, M) = \sum_{l\,:\,C_l \in M} w(C_k, C_l). \tag{3.3}$$

Suppose now that we single out one component C_k whose framing we want to change. Then we can rewrite the right-hand side of (1.10) as follows:

$$\begin{aligned}
&(t^{1/2}+t^{-1/2})\sum_{J\subset L}(-1)^{|J|}t^{3f(J^c)-6n(J)}V_{\widetilde{J}}(t)\\
&\quad= (t^{1/2}+t^{-1/2})\sum_{M\subset L\setminus C_k}(-1)^{|M|}\{t^{3f(\setminus M)-6n(M)}V_{\widetilde{M}}(t)\\
&\qquad\qquad - t^{3f(L\setminus M)-3f(C_k)-6n(M\cup C_k)}V_{\widetilde{M}\cup\widetilde{C}_k}(t)\}\\
&\quad= (t^{1/2}+t^{-1/2})\sum_{M\subset L\setminus C_k}(-1)^{|M|}t^{3f(L\setminus M)-6n(M)}\\
&\qquad\qquad \times\{V_{\widetilde{M}}(t) - t^{-3f(C_k)-6w(C_k, M)}V_{\widetilde{M}\cup\widetilde{C}_k}(t)\}.
\end{aligned} \tag{3.4}$$

Now we can use the relation (2.2) to see what happens when the framing on C_k is changed. Suppose that the two strands in $\widetilde{C}_k$ are twisted as shown in Figure 3.1(a).



(a) (b) (c)

FIGURE 3.1. Changing framing by one

By applying (2.2), we get a relation between the Jones polynomials of the three links shown in Figure 3.1. Clearly the framing in (b) is one less than in (a). Also, since $\widetilde{C}_k$ is composed of two parallel strands, the diagram in (c) collapses down to the unknot. Let us denote by $\widetilde{C}'_k$ the two strands shown in Figure 3.1(b), whose linking number is $f(C_k) - 1$. Applying (2.2) gives

$$V_{\widetilde{M}\cup\widetilde{C}_k}(t) = tV_{\widetilde{M}\cup\widetilde{C}'_k}(t) + (1 - t^{-2})t^{3f(C_k)+6w(C_k, M)}V_{\widetilde{M}}(t). \tag{3.5}$$

Substituting (3.5) into (3.4) gives

$$\begin{aligned}(t^{1/2} + t^{-1/2})t \sum_{M\subset L\backslash C_k} &(-1)^{|M|}t^{3f(L\backslash M)-6n(M)} \\ &\times \{V_{\widetilde{M}}(t) - t^{-3[f(C_k)-1]-6w(C_k, M)}V_{\widetilde{M}\cup\widetilde{C}'_k}(t)\}.\end{aligned} \tag{3.6}$$

Apart from the overall factor of t, (3.6) is the right-hand side of (1.10) corresponding to the framed link L with the framing of C_k reduced by one. Furthermore, the left-hand side of (1.10) also changes by a factor of t when $f(C_k)$ is reduced by one. Therefore if (1.10) holds for any particular choice of framing, it also holds for all other framings.

We now turn to the proof of Theorem 2. We will choose a particular framing which simplifies the proof, and then appeal to the previous argument to derive the general result. We will consider separately the two cases distinguished earlier.

Case 1. We will use the same notation introduced for Case 1 in §2. We choose the framings f_L and f_R to be zero, and also choose each component of M to have zero framing. We can now compute the following relations between linking numbers:

$$\begin{aligned} n(L_+) = n(L_-) &= n(M) + w_L(M) + w_R(M), \\ n(L_0) &= n(M) + w_L(M) + w_R(M) + \alpha, \\ n(L_\infty) &= n(M) + w_L(M) - w_R(M). \end{aligned} \tag{3.7}$$

Similar relations hold if M is replaced by any sublink J of M, and if L_+ is replaced by $J \cup C_+$, etc. We will now write down the four expressions which (1.10) provides for the four links L_+, L_-, L_0 and L_∞. We again collect terms together in the manner of (3.4). The left-hand side will be denoted F_+, F_-, F_0 and F_∞ respectively. For convenience, we will write g instead of $(t^{1/2}+t^{-1/2})(1+t+t^{-1})^{-1}$.

$$\begin{aligned}(A_1)\qquad & t^{1+2\alpha-4n(M)-4w_L(M)-4w_R(M)}F_+\\ &= g\sum_{J\subset M}(-1)^{|J|}\{t^{3+6\alpha-6n(J)}V_{\widetilde{J}}(t)\\ &\qquad - t^{-6n(J)-6w_L(J)-6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_+}(t)\}.\end{aligned}$$

$$\begin{aligned}(A_2)\qquad & t^{-1+2\alpha-4n(M)-4w_L(M)-4w_R(M)}F_-\\ &= g\sum_{J\subset M}(-1)^{|J|}\{t^{-3+6\alpha-6n(J)}V_{\widetilde{J}}(t)\\ &\qquad - t^{-6n(J)-6w_L(J)-6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_-}(t)\}.\end{aligned}$$

$$\begin{aligned}(A_3)\qquad & t^{-4n(M)-4w_L(M)-4w_R(M)-4\alpha}F_0\\ &= g\sum_{J\subset M}(-1)^{|J|}\{t^{-6n(J)}V_{\widetilde{J}}(t) - t^{-6n(J)-6w_L(J)}V_{\widetilde{J}\cup\widetilde{C}_L}(t)\\ &\qquad - t^{-6n(J)-6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_R}(t)\\ &\qquad + t^{-6n(J)-6w_L(J)-6w_R(J)-6\alpha}V_{\widetilde{J}\cup\widetilde{C}_L\cup\widetilde{C}_R}(t)\}.\end{aligned}$$

$$\begin{aligned}(A_4)\qquad & t^{-2\alpha-4n(M)-4w_L(M)+4w_R(M)}F_\infty\\ &= g\sum_{J\subset M}(-1)^{|J|}\{t^{-6\alpha-6n(J)}V_{\widetilde{J}}(t)\\ &\qquad - t^{-6n(J)-6w_L(J)+6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_\infty}(t)\}.\end{aligned}$$

Our strategy now is to show that F_+, F_-, F_0 and F_∞ satisfy the crossing relation (3.1). In order to do this, we consider the following combination of expressions (A_1) and (A_2):

$$\begin{aligned}(3.8)\qquad t^{-3}A_1 - t^3A_2 = -g\sum_{J\subset M}&(-1)^{|J|}t^{-6n(J)-6w_L(J)-6w_R(J)}\\ &\times\{t^{-3}V_{\widetilde{J}\cup\widetilde{C}_+}(t) - t^3V_{\widetilde{J}\cup\widetilde{C}_-}(t)\}.\end{aligned}$$

Now we use the result of Proposition 2.1, applying it to $J\cup C_+$ and $J\cup C_-$. Recall that f_L and f_R are zero.

$$\begin{aligned}(3.9)\qquad t^{-3}A_1 - t^3A_2 = (t-t^{-1})g\sum_{J\subset M}&(-1)^{|J|}t^{-6n(J)-6w_L(J)-6w_R(J)}\\ &\times\{V_{\widetilde{J}\cup\widetilde{C}_L\cup\widetilde{C}_R}(t) + 2t^{6\alpha+6w_L(J)+6w_R(J)}V_{\widetilde{J}}(t)\\ &\qquad - t^{12\alpha+12w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_\infty}(t) - t^{6\alpha+6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_L}(t)\\ &\qquad - t^{6\alpha+6w_L(J)}V_{\widetilde{J}\cup\widetilde{C}_R}(t)\}.\end{aligned}$$

Combining expressions (A_3) and (A_4), we also have

$$\begin{aligned} &t^{6\alpha}A_3 + t^{12\alpha}A_4 \\ &\quad = g\sum_{J\subset M}(-1)^{|J|}t^{-6n(J)}\{2t^{6\alpha}V_{\widetilde{J}}(t) - t^{6\alpha-6w_L(J)}V_{\widetilde{J}\cup\widetilde{C}_L}(t) \\ &\qquad - t^{6\alpha-6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_R}(t) \\ &\qquad + t^{-6w_L(J)-6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_L\cup\widetilde{C}_R}(t) \\ &\qquad - t^{12\alpha-6w_L(J)+6w_R(J)}V_{\widetilde{J}\cup\widetilde{C}_\infty}(t)\}. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we deduce that

$$t^{-3}A_1 - t^3A_2 = (t-t^{-1})(t^{6\alpha}A_3 + t^{12\alpha}A_4). \tag{3.11}$$

Inserting the left sides of the expressions into (3.11), we deduce the relation

$$\begin{aligned} &t^{2\alpha-4n(M)-4w_L(M)-4w_R(M)}\{t^{-2}F_+ - t^2F_-\} \\ &\quad = (t-t^{-1})t^{2\alpha-4n(M)-4w_L(M)-4w_R(M)}\{F_0 + t^{8\alpha+8w_R(M)}F_\infty\}. \end{aligned} \tag{3.12}$$

Now recall that in (3.1), λ is the linking number of C_R with $M\cup C_L$, and this is $\alpha + w_R(M)$. Therefore if we define

$$a = it^{-2}, \qquad x = i(t-t^{-1}), \tag{3.13}$$

we deduce from (3.12) that

$$aF_+ + a^{-1}F_- = x(F_0 + a^{-4\lambda}F_\infty),$$

which is the desired result.

Case 2. Again we use the notation described in §2 for Case 2. Now we will choose f_1 and f_2 to be zero, and also we choose zero framing for every component in M. The following relations now follow from explicit computations:

$$\begin{aligned} n(L_+) &= n(M) + w_1(M) + w_2(M) + \beta, \\ n(L_-) &= n(M) + w_1(M) + w_2(M) + \beta - 1, \\ n(L_0) &= n(M) + w_1(M) + w_2(M), \\ n(L_\infty) &= n(M) - w_1(M) + w_2(M). \end{aligned} \tag{3.14}$$

We can now write down the expressions which (1.10) provides for the links L_+, L_-, L_0, L_∞,

$$(B_1)\quad \begin{aligned} &t^{-4n(M)-4w_1(M)-4w_2(M)-4\beta}F_+ \\ &\quad = g\sum_{J\subset M}(-1)^{|J|}\{t^{-6n(J)}V_{\widetilde{J}}(t) - t^{-6n(J)-6w_1(J)}V_{\widetilde{J}\cup\widetilde{C}_1}(t) \\ &\qquad - t^{-6n(J)-6w_2(J)}V_{\widetilde{J}\cup\widetilde{C}_2}(t) \\ &\qquad + t^{-6n(J)-6w_1(J)-6w_2(J)-6\beta}V_{\widetilde{J}\cup\widetilde{C}_+}(t)\}. \end{aligned}$$

$$(B_2)\quad \begin{aligned} &t^{-4n(M)-4w_1(M)-4w_2(M)-4\beta+4}F_- \\ &\quad = g\sum_{J\subset M}(-1)^{|J|}\{t^{-6n(J)}V_{\widetilde{J}}(t) - t^{-6n(J)-6w_1(J)}V_{\widetilde{J}\cup\widetilde{C}_1}(t) \\ &\qquad - t^{-6n(J)-6w_2(J)}V_{\widetilde{J}\cup\widetilde{C}_2}(t) \\ &\qquad + t^{-6n(J)-6w_1(J)-6w_2(J)-6\beta+6}V_{\widetilde{J}\cup\widetilde{C}_-}(t)\}. \end{aligned}$$

$$(B_3)\quad \begin{aligned} &t^{2\beta-1-4n(M)-4w_1(M)-4w_2(M)}F_0 \\ &\quad = g\sum_{J\subset M}(-1)^{|J|}\{t^{6\beta-3-6n(J)}V_{\widetilde{J}}(t) \\ &\qquad - t^{-6n(J)-6w_1(J)-6w_2(J)}V_{\widetilde{J}\cup\widetilde{C}_0}(t)\}. \end{aligned}$$

$$(B_4)\quad \begin{aligned} &t^{-2\beta+1-4n(M)+4w_1(M)-4w_2(M)}F_\infty \\ &\quad = g\sum_{J\subset M}(-1)^{|J|}\{t^{-6\beta+3-6n(J)}V_{\widetilde{J}}(t) \\ &\qquad - t^{-6n(J)+6w_1(J)-6w_2(J)}V_{\widetilde{J}\cup\widetilde{C}_\infty}(t)\}. \end{aligned}$$

Taking linear combinations, we deduce

$$(3.15)\quad \begin{aligned} B_1 - B_2 = g\sum_{J\subset M}&(-1)^{|J|}t^{-6n(J)-6w_1(J)-6w_2(J)-6\beta+3} \\ &\times\{t^{-3}V_{\widetilde{J}\cup\widetilde{C}_+}(t) - t^3V_{\widetilde{J}\cup\widetilde{C}_-}(t)\}. \end{aligned}$$

Again using the result of Proposition 2.2, we get

$$(3.16)\quad \begin{aligned} B_1 - B_2 = (t-t^{-1})g\sum_{J\subset M}&(-1)^{|J|}t^{-6\beta+3-6n(J)-6w_1(J)-6w_2(J)} \\ &\times\{t^{12\beta-6+12w_1(J)}V_{\widetilde{J}\cup\widetilde{C}_\infty}(t) - V_{\widetilde{J}\cup\widetilde{C}_0}(t)\}. \end{aligned}$$

Furthermore, we also have the relation

$$(3.17)\quad \begin{aligned} &t^{-6\beta+3}B_3 - t^{6\beta-3}B_4 \\ &\quad = g\sum_{J\subset M}(-1)^{|J|}\{t^{6\beta-3-6n(J)+6w_1(J)-6w_2(J)}V_{\widetilde{J}\cup\widetilde{C}_\infty}(t) \\ &\qquad - t^{-6\beta+3-6n(J)-6w_1(J)-6w_2(J)}V_{\widetilde{J}\cup\widetilde{C}_0}(t)\}. \end{aligned}$$

Combining (3.16) and (3.17) we deduce

$$(3.18)\quad B_1 - B_2 = (t-t^{-1})(t^{-6\beta+3}B_3 - t^{6\beta-3}B_4).$$

Next we substitute the left-hand sides of the expressions into (3.18). After some cancellation this reduces to

$$t^{-2}F_+ - t^2F_- = (t - t^{-1})\{F_0 - t^{8\beta+8w_1(M)-4}F_\infty\}. \tag{3.19}$$

Comparing this with (3.2), we see that μ is the linking number of C_1 with $M \cup C_2$, and this is $\beta + w_1(M)$. Therefore defining a and x by (3.13), we get

$$aF_+ + a^{-1}F_- = x(F_0 + a^{-4\mu+2}F_\infty).$$

Therefore we have verified the crossing relations for F_L in the expression (1.10). From the first example given in the introduction, we know that F_U is one, where U is the unknot. Therefore F_L is the Kauffman polynomial.

Finally we prove Theorem 3. This follows by substituting the right-hand side of (1.10) into (1.18). This gives

$$\begin{aligned}(t^{1/2}+t^{-1/2})&\sum_{J\subset L}(-1)^{|J|}t^{3f(J^c)}\sum_{M\subset J}(-1)^{|M|}t^{3f(J\setminus M)-6n(M)}V_{\tilde{M}}(t)\\ &= (t^{1/2}+t^{-1/2})\sum_{J\subset L}(-1)^{|J|}\sum_{M\subset J}(-1)^{|M|}t^{3f(L\setminus M)-6n(M)}V_{\tilde{M}}(t)\\ &= (t^{1/2}+t^{-1/2})\sum_{M\subset L}(-1)^{|M|}t^{3f(M^c)-6n(M)}V_{\tilde{M}}(t)\sum_{N\subset L\setminus M}(-1)^{|N|+|M|}.\end{aligned} \tag{3.20}$$

However we have the identity

$$\sum_{N\subset L\setminus M}(-1)^{|N|} = 0, \tag{3.21}$$

unless $L = M$, in which case (3.21) is one. Substituting into (3.20) yields the left-hand side of (1.18), and this proves Theorem 3.

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